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Thème : § 1-3 Résolution analytique d'une équation différentielle ordinaire du premier ordre
Lien vers les énoncés des travaux dirigés:
https://www.deleze.name/marcel/sec2/applmaths/csud/eq-differentielles/1-3_EQ-DIFFERENTIELLES.pdf

# § 1.3 TD 1 Corrigé

## a)

Méthode de la séparation des variables

$$\frac{dN}{dt} = -\lambda N$$

$$\frac{dN}{N} = -\lambda\, dt$$

$$\int \frac{1}{N}\, dN = -\lambda \int dt$$

$$\ln |N(t)| = -\lambda t + c$$

$$|N(t)| = e^{-\lambda t + c} = e^{c}\, e^{-\lambda t}$$

$N(t) = k\, e^{-\lambda t}$ où $k = \pm e^{c}$ est une constante réelle

$$N(0) = k = N_0$$

$$\boxed{N(t) = N_0\, e^{-\lambda t}}$$

Avec *Mathematica*

```
Clear[n, t];
```
efface

```
DSolve[{n'[t] == -λ n[t], n[0] == n0}, n[t], t]
```
résous équation différentiel

```
{{n[t] → e^(-t λ) n0}}
```

## b)

T = demi-vie

$$N(T) = \frac{1}{2} N_0$$

$$N_0\, e^{-\lambda T} = \frac{1}{2} N_0$$

$$e^{-\lambda T} = \frac{1}{2}$$

$$-\lambda T = \ln\left(\frac{1}{2}\right) = -\ln(2)$$

$$\boxed{\lambda T = \ln(2)}$$

## c)

$\tau$ = vie moyenne

$$N(\tau) = \frac{1}{e} N_0$$

$$N_0 e^{-\lambda \tau} = \frac{1}{e} N_0$$

$$e^{-\lambda \tau} = e^{-1}$$

$$-\lambda \tau = -1$$

$$\boxed{\lambda \tau = 1}$$

$$\lambda = \frac{\ln(2)}{T} = \frac{1}{\tau}$$

$$\boxed{T = \tau \ln(2)}$$

## d)

Activité A(t)

$$A(t) = -\frac{d}{dt} N(t) = -\frac{d}{dt}\left(N_0 e^{-\lambda t}\right) = (-1) N_0 (-\lambda) e^{-\lambda t} = \lambda N_0 e^{-\lambda t}$$

$$A(0) = \lambda N_0 = A_0 = \text{activité initiale}$$

$$\boxed{A(t) = A_0 e^{-\lambda t}}$$

$$A(t) = 2000 e^{-\lambda t}$$

$$A(90\,\text{min}) = 630 \quad \Longrightarrow \quad 2000 e^{-\lambda (90\,\text{min})} = 630$$

$$e^{-\lambda (90\,\text{min})} = \frac{630}{2000} \quad \Longrightarrow \quad -\lambda (90\,\text{min}) = \ln \frac{63}{200}$$

Constante de désintégration $\lambda$

$$\lambda = \frac{-1}{90\,\text{min}} \ln \frac{63}{200} = \frac{1}{90} \ln \frac{200}{63} \text{min}^{-1} \simeq 0.0128\,\text{min}^{-1}$$

Vie moyenne $\tau$

$$\tau = \frac{1}{\lambda} \simeq 77.9\,\text{min}$$

Demi-vie T

$$T = \frac{\ln(2)}{\lambda} = \tau \ln(2) \simeq 54.0\,\text{min}$$

## e)

Activité A(t)

$$A(t) = A_0 e^{-\lambda t}$$

$$T = 5568\,\text{ans} \quad \Longrightarrow \quad \lambda = \frac{\ln(2)}{T} = \frac{\ln(2)}{5568\,\text{an}}$$

$$A(0) \simeq 6.68 \qquad (t = 0 \quad \text{à la mort de l'arbre})$$

$$A(t) \simeq 0.97 \qquad (t = \text{nombre d'années après la mort de l'arbre} = \text{durée})$$

$$0.97 = 6.68 e^{-\lambda t}$$

$$e^{-\lambda t} = \frac{0.97}{6.68}$$

$$\lambda t = -\ln\left(\frac{0.97}{6.68}\right) = \ln\left(\frac{6.68}{0.97}\right)$$

$$\frac{\ln(2)}{5568\text{ an}} t = \ln\left(\frac{6.68}{0.97}\right)$$

$$t = \frac{5568\text{ an}}{\ln(2)} \ln\left(\frac{6.68}{0.97}\right) \simeq 15 \times 500\text{ an}$$

Date

$$t - 1950\text{ ans} \simeq 13'550\text{ ans avant J. C.}$$

# § 1.3 TD 2 Corrigé

## a)

Méthode de la séparation des variables

$$\frac{dp}{dz} = -p\frac{M g}{R T}$$

$$\frac{dp}{p} = -\frac{M g}{R T} dz$$

$$\int \frac{1}{p} dp = -\frac{M g}{R T}\int dz$$

$$\ln|p(z)| = -\frac{M g}{R T} z + c$$

$$|p(z)| = e^{-\frac{M g}{R T} z + c} = e^c e^{-\frac{M g}{R T} z}$$

$p(z) = k e^{-\frac{M g}{R T} z}$ où $k = \pm e^c$ est une constante réelle

$$p(0) = k = p_0$$

$$\boxed{p(z) = p_0 \, e^{-\frac{M g}{R T} z}}$$

Avec *Mathematica*

```
Clear[p, z];
```
efface

```
DSolve[{p'[z] == -(M g)/(R T) p[z], p[0] == p0}, p[z], z]
```
résous équation différentielle

$$\left\{\left\{p[z] \to e^{-\frac{g M z}{R T}} p0\right\}\right\}$$

# § 1.3 TD 3 Corrigé

## 1°

Décomposition en fractions simples : pour tout *p*,

$$\frac{1}{a p - b p^2} = \frac{1}{p(a - b p)} = \frac{A}{p} + \frac{B}{a - b p} = \frac{A(a - b p) + B p}{p(a - b p)} = \frac{(-A b + B) p + (A a)}{p(a - b p)}$$

$$\Longrightarrow \begin{cases} -A b + B = 0 \\ A a = 1 \end{cases}$$

$$\Longrightarrow \begin{cases} A = \frac{1}{a} \\ B = A\,b = \frac{b}{a} \end{cases}$$

$$\boxed{\frac{1}{a\,p - b\,p^2} = \frac{1}{a}\left(\frac{1}{p} + \frac{b}{a - b\,p}\right)}$$

## 2°

Séparation des variables

$$\frac{dp}{dt} = a\,p - b\,p^2$$

$$\frac{dp}{a\,p - b\,p^2} = dt$$

$$\int \frac{1}{a\,p - b\,p^2}\,dp = \int 1\,dt$$

$$\frac{1}{a}\left(\int \frac{1}{p}\,dp + b\int \frac{1}{a - b\,p}\,dp\right) = \int 1\,dt$$

$$\frac{1}{a}\left(\ln|p| + \frac{b}{-b}\ln|a - b\,p|\right) = t + c_1$$

$$\ln\left|\frac{p}{a - b\,p}\right| = a\,t + c_2$$

$$\left|\frac{p(t)}{a - b\,p(t)}\right| = e^{a\,t + c_2} = e^{c_2}\,e^{a\,t}$$

$$\frac{p(t)}{a - b\,p(t)} = k\,e^{a\,t} \qquad \text{où} \qquad k = \pm e^{c_2}$$

$$p(t) = k\,e^{a\,t}\,(a - b\,p(t)) = a\,k\,e^{a\,t} - b\,k\,e^{a\,t}\,p(t)$$

$$\text{où} \qquad p(0) = k\,1\,(a - b\,p(0)) \qquad \text{d'où} \qquad k = \frac{p_0}{a - b\,p_0}$$

$$p(t) = \frac{p_0}{a - b\,p_0}\,e^{a\,t}\,(a - b\,p(t))$$

$$(a - b\,p_0)\,p(t) = a\,p_0\,e^{a\,t} - b\,p_0\,e^{a\,t}\,p(t))$$

$$\left((a - b\,p_0) + b\,p_0\,e^{a\,t}\right)\,p(t) = a\,p_0\,e^{a\,t}$$

$$\boxed{p(t) = \frac{a\,p_0\,e^{a\,t}}{a - b\,p_0 + b\,p_0\,e^{a\,t}}}$$

$$= \frac{\frac{a}{b}}{\frac{a}{b\,p_0}\,e^{-a\,t} - e^{-a\,t} + 1} = \frac{L}{1 + \left(\frac{L}{p_0} - 1\right)\,e^{-a\,t}} \qquad \text{où} \qquad L = \frac{a}{b}$$

Avec *Mathematica*

```
Clear[p, t, a, b];
```
efface

```
DSolve[{p'[t] == a p[t] - b p[t]^2, p[0] == p0}, p[t], t]
```
résous équation différentiel

... Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$$\left\{\left\{p[t] \to \frac{a\, e^{a t}\, p0}{a - b\, p0 + b\, e^{a t}\, p0}\right\}\right\}$$

## 3°

$p_\infty =$

$$\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \frac{a\, p_0\, e^{a t}}{a - b\, p_0 + b\, p_0\, e^{a t}} = \lim_{t\to\infty} \frac{a\, p_0}{(a - b\, p_0)\, e^{-a t} + b\, p_0} = \frac{a\, p_0}{0 + b\, p_0} = \frac{a}{b} \simeq 10 \times 10^9$$

## 4°

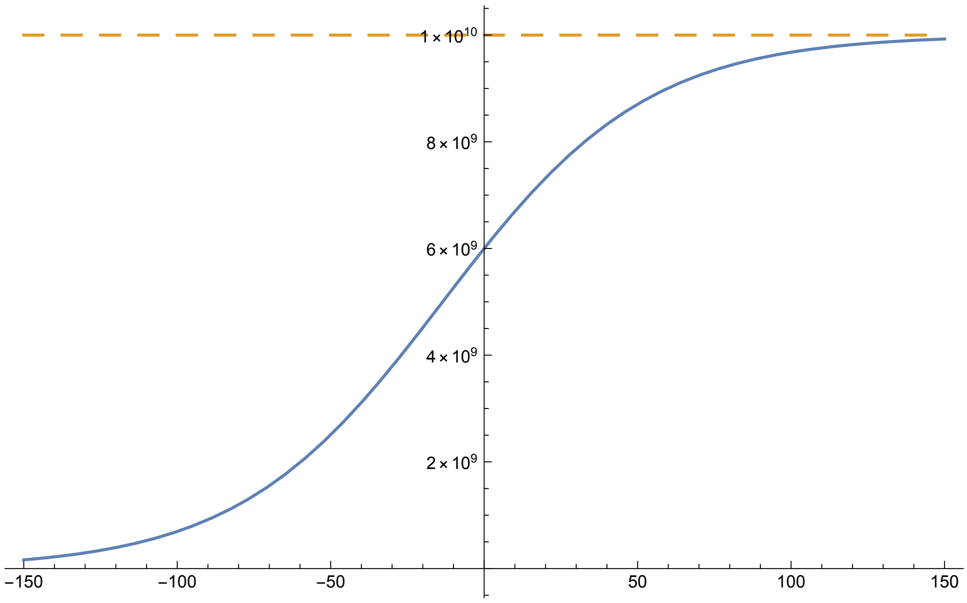
$$p[t\_] := \frac{a\, p0\, e^{a t}}{a - b\, p0 + b\, p0\, e^{a t}} \; /. \; \{a \to 0.03,\ b \to 3 \times 10^{-12},\ p0 \to 6 \times 10^9\}$$

```
p[50]
```

$8.70509 \times 10^9$

$$\text{Plot}\left[\left\{p[t], \frac{a}{b} \; /. \; \{a \to 0.03,\ b \to 3 \times 10^{-12},\ p0 \to 6 \times 10^9\}\right\}, \{t, -150, 150\},\right.$$
tracé de courbes

$$\left.\text{PlotStyle} \to \{\text{Dashing}[\{\}], \text{Dashing}[\{0.02\}]\},\ \text{ImageSize} \to \{500, 350\}\right]$$
style de tracé | style de rayures | style de rayures | taille d'image




# 1.3- TD 4 Facultatif

Avec les notations de *Mathematica*

$$\frac{\sqrt{p}\ \text{Tanh}\left[\frac{\sqrt{p}\ \sqrt{r}\ \text{Log}[m0]-\sqrt{p}\ \sqrt{r}\ \text{Log}[m0-q\,t]}{q}\right]}{\sqrt{r}} = \frac{\sqrt{4}\ \text{Tanh}\left[\frac{\sqrt{4}\ \sqrt{1}\ \text{Log}[8]-\sqrt{4}\ \sqrt{1}\ \text{Log}[8-2\,t]}{2}\right]}{\sqrt{1}}$$

Avec les notations usuelles en mathématiques

$$s(t) = 2\tanh\left(\frac{2\ln(8) - 2\ln(8-2t)}{2}\right) = 2\tanh(\ln(8) - \ln(8-2t)) =$$

$$2\tanh\left(\ln\left(\frac{8}{8-2t}\right)\right) = 2\tanh\left(\ln\left(\frac{4}{4-t}\right)\right) = 2\,\frac{e^{\ln\left(\frac{4}{4-t}\right)} - e^{-\ln\left(\frac{4}{4-t}\right)}}{e^{\ln\left(\frac{4}{4-t}\right)} + e^{-\ln\left(\frac{4}{4-t}\right)}} =$$

$$2\,\frac{e^{\ln\left(\frac{4}{4-t}\right)} - e^{\ln\left(\frac{4-t}{4}\right)}}{e^{\ln\left(\frac{4}{4-t}\right)} + e^{\ln\left(\frac{4-t}{4}\right)}} = 2\,\frac{\frac{4}{4-t} - \frac{4-t}{4}}{\frac{4}{4-t} + \frac{4-t}{4}} = 2\,\frac{16 - (4-t)^2}{16 + (4-t)^2} = 2\,\frac{-t^2 + 8t}{t^2 - 8t + 32}$$