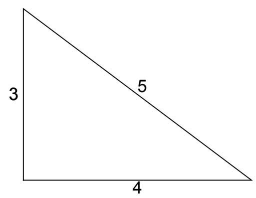
Demonstration of "$\sqrt{2}$ is irrational"

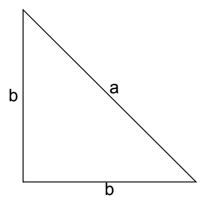
An example of a demonstration by the absurd

Introduction

We can construct a right-angled triangle whose three sides are whole numbers:



Is it possible to construct a square whose side b and diagonal a are both measured by integers whole numbers? (The figure below shows half of the square, which is an isosceles right-angled triangle).



According to the Pythagorean theorem

$$\begin{aligned} b^2 + b^2 &= a^2 \\ 2b^2 &= a^2 \\ 2 &= \left(\frac{a}{b}\right)^2 \\ \frac{a}{b} &= \sqrt{2} \end{aligned}$$

If it's possible to find integers a, b that verify this equality, we say that $\sqrt{2}$ is a rational number. Otherwise we say that $\sqrt{2}$ is an irrational number.

Lemma

For any integer a, if a^2 is even, then a is even.

Demonstration by contraposition : Let's show that, if a is odd, then a^2 is odd. Let $a = 2n + 1$. Then $a^2 = (2n + 1)^2 = 4n^2 + 4n + 1$, which is odd.

Demonstration of "$\sqrt{2}$ is irrational".

Let's assume for the sake of argument that $\sqrt{2}$ is rational: then $\sqrt{2} = \frac{a}{b}$ where a, b are positive integers. It is possible to simplify the fraction $\frac{a}{b}$ until a, b are prime to each other (i.e. the fraction $\frac{a}{b}$ can no longer be simplified).

$$\sqrt{2} = \frac{a}{b}$$
$$\sqrt{2}b = a$$
$$2b^2 = a^2$$

Since a^2 is even, a is even and $a = 2p$ where p is a positive integer.

$$2b^2 = (2p)^2$$
$$2b^2 = 4p^2$$
$$b^2 = 2p^2$$

Since b^2 is even, b is even. Consequently, it is possible to simplify the fraction $\frac{a}{b}$ by 2, which contradicts the assumption that a, b are mutually prime.

Since the hypothesis "$\sqrt{2}$ is rational" leads to a contradiction, the opposite is true, i.e. "$\sqrt{2}$ is irrational".

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Link to parent page: Marcel Délèze website

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